Greek Letters

β = crystallization parameter defined by Kane et al. (1974) (s^{-1})
θ = elapsed time after seeding at which crystal was formed (min.)
λ = latent specific heat of ice (cal/g)
ρ_I = density of ice (g/cm^3)
ρ_s = weight concentration of the solute (g/cm^3)
ρ_s'' = equilibrium weight concentration of solute at ice surface (g/cm^3)
ω = crystal aspect ratio, that is, thickness to diameter ratio

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Optimal Feedback Control of Time-Delay Systems

The suboptimal control policy obtained from series expansion of time-delay terms is useful when the delays are small. The second method involving direct search on the constant gain matrix is applicable even for large delays. Both of these proposed methods are computationally simple and lead to easy implementation in practice.

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SCOPE

Engineering systems often contain delayed elements such as recycle streams, transportation lags, and time-delayed feedback signals. In particular, such delayed elements are commonly encountered in the chemical and petroleum industries, and, for proper control, such time lags cannot be neglected. The dynamic behavior of such time-delay systems can be modeled adequately by differential-difference equations, that is, differential equations with delayed arguments.

In contrast to the case with no time delay, the optimal control theory for time-delay systems has not yet been extensively developed; it is only recently that the subject has received considerable attention in the literature. Since the work in this area was pioneered by Kharatishvili (1961) who extended Pontryagin's maximum principle to a system with a single delay in the state variable, numerous publications have appeared on the derivation of necessary conditions for optimality of time-delay systems and on the computational methods for solving the resulting difficult two-point boundary value problems. However, relatively few authors have discussed the synthesis of optimal feedback control for time-delay systems (Eller et al., 1969; Ross, 1971; Soliman and Ray, 1972). The computations involved in the existing methods, however, are rather complicated, and the resulting feedback controllers are not readily suited for implementation in practice.

The main objective of this study is to present two simple methods for designing practical suboptimal feedback controllers for systems with delayed states and a quadratic performance index. The first approach consists of approximating delayed states by Taylor series expansion and applying to the resulting approximate nondelay model the well-developed optimal control theory for lumped parameter systems. The second method involves employing the direct search optimization technique to determine the best time-invariant feedback gain matrix based on the control structure which includes the present state and some time-delay terms. The proposed methods are tested and evaluated by using both linear and nonlinear examples.

CONCLUSIONS AND SIGNIFICANCE

The present study shows that the two design approaches proposed here lead to good suboptimal feedback control of time-delay systems with a quadratic performance index. Although the approach based on Taylor series expansion of delayed state has the advantage of retaining sufficient dynamic features of the original time-delay system when time delays are sufficiently small, the direct search on the constant feedback gains based on a simple feedback control structure yields good results for a considerably wider range of time delays. Furthermore, when the final time is relatively large compared to the time delay, the direct search approach provides an excellent suboptimal feedback control for the time-delay system, making this design method particularly useful in the optimal control of industrial processes.

The attractive features of the proposed design methods include computational simplicity, ease of implementation of the resulting control systems, and the low degree of suboptimality. In addition, these design approaches are applicable to nonlinear time-delay systems.

The optimal feedback control problems for systems with pure time delays are of considerable practical importance. Because of the complexity of the problem, most of the work in this area has been devoted to linear systems with a quadratic performance index, which yield particularly useful optimal feedback controllers.

The optimal feedback control of linear-quadratic systems with delays in the state variables was first considered by Krasovskii (1962, 1963). Ross and Flügge-Lotz (1969) derived an optimal feedback control law for a class of time-invariant linear systems with infinite final time and a single delay in the state variables. Later, Ross (1971) extended these results to the case with multiple delays in the state. The optimal feedback control of linear time varying systems with delayed states was considered by Eller et al. (1969). Recently, Soliman and Ray (1972) derived the general optimal feedback control law for multiple time varying delays in both the state and the control variables. Their approach consists of transforming the original differential-difference model into an arbitrarily large number of ordinary differential equations through the use of discretization and applying the well-known classical linear-quadratic optimal control theory for nondelay systems.

However, the actual implementation of the optimal feedback control system based on the results of these investigations presents some difficulties, since the off-line computation of control system parameters requires considerable effort, and also the on-line computations and storage requirements are significant. Therefore, the purpose of this paper is to present two simple and straightforward design methods for the suboptimal feedback control of time-delay systems which lead to easy implementation of the resulting control systems in practice. Of particular interest here is the system with delayed states and a quadratic performance index.

THEORETICAL BACKGROUND

Consider a linear time-delay system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\theta) + \mathbf{C}\mathbf{u}(t) \tag{1}$$

with the given initial state profile

$$\mathbf{x}(\tau) = \boldsymbol{\varphi}(\tau), \quad -\theta \leq \tau \leq 0 \tag{2}$$

and the performance index

$$J = \frac{1}{2}\int_0^{t_f} [\mathbf{x}^T(t)\,\mathbf{Q}\,\mathbf{x}(t) + \mathbf{u}^T(t)\,\mathbf{R}\,\mathbf{u}(t)]\,dt \tag{3}$$

where $\mathbf{Q}$ is a constant symmetric positive semidefinite matrix, and $\mathbf{R}$ is a constant symmetric positive definite matrix. The problem is to find the control policy $\mathbf{u}(t)$, $0 \leq t \leq t_f$, which will minimize the performance index of Equation (3) for the system described by Equations (1) and (2).

Recently, Mutharasan and Luus (1975) showed that Taylor series expansion can be successfully used for the analysis of time-delay systems. Therefore, the first proposed method in this paper involves expanding the delayed state variable $\mathbf{x}(t-\theta)$ in terms of a Taylor series and retaining only the first two terms.

That is

$$\mathbf{x}(t-\theta) \simeq \mathbf{x}(t) - \theta\,\dot{\mathbf{x}}(t) \tag{4}$$

Substitution of Equation (4) into Equation (1) yields

$$\dot{\mathbf{x}}(t) = \mathbf{M}\,\mathbf{x}(t) + \mathbf{N}\,\mathbf{u}(t) \tag{5}$$

where

$$\mathbf{M} = (\mathbf{I} + \theta\mathbf{B})^{-1}\,(\mathbf{A} + \mathbf{B}) \tag{6}$$

$$\mathbf{N} = (\mathbf{I} + \theta\mathbf{B})^{-1}\,\mathbf{C} \tag{7}$$

The corresponding initial condition is

$$\mathbf{x}(0) = \boldsymbol{\varphi}(0) \tag{8}$$

It should be noted that the original problem has been reduced through the approximation of Equation (4) to the problem of optimal control of a linear nondelay system with a quadratic performance index. The optimal control policy for the approximate model described by Equations (5) to (8) with the performance index of Equation (3) can be readily determined from the well-developed theory on nondelay systems (for example, Athans and Falb, 1966; Lapidus and Luus, 1967) as

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{N}^T\mathbf{K}(t)\,\mathbf{x}(t) \tag{9}$$

where the time-varying feedback gain matrix $\mathbf{K}(t)$ is generated by solving the Riccati equation

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\,\mathbf{M} - \mathbf{M}^T\mathbf{K}(t) + \mathbf{K}(t)\,\mathbf{N}\mathbf{R}^{-1}\mathbf{N}^T\mathbf{K}(t) - \mathbf{Q} \tag{10}$$

with the final condition

$$\mathbf{K}(t_f) = 0 \tag{11}$$

The control policy determined from Equations (9), (10), and (11) is expected to become closer to the optimal control policy for the original time-delay system as the magnitude of time delay is decreased. However, the degree of suboptimality of the proposed method for a given time delay cannot be determined a priori, since it depends on the dynamics of a specific problem under consideration. Also, it is obvious that this approach may be applied equally well to the case with multiple delays.

The second approach is based on the optimal feedback control law derived by Ross and Flügge-Lotz (1969) for a linear time-delay system with infinite final time. They showed that the optimal control law for the system described by Equations (1), (2), and (3) with $t_f = \infty$ takes the form

$$\mathbf{u}(t) = \mathbf{K}_0\,\mathbf{x}(t) + \int_{-\theta}^{0} \mathbf{K}_1(\eta)\,\mathbf{x}(t+\eta)\,d\eta \tag{12}$$

where $\mathbf{K}_0$ is a constant matrix. When $\mathbf{x}(t+\eta)$ in the above integral is approximated by n straight-line segments, Equation (12) becomes

$$\mathbf{u}(t) = \mathbf{G}_0\,\mathbf{x}(t) + \sum_{i=1}^{n} \mathbf{G}_i\,\mathbf{x}\left(t - \frac{i\theta}{n}\right) \tag{13}$$

where $\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_n$ are constant gain matrices. The second design approach proposed in this paper, then, consists of choosing through optimization the best elements of constant gain matrices $\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_n$. For optimization, we shall use the direct search optimization procedure of Luus and Jaakola (1973) which is well suited for the problem, and therefore we refer to this method as the direct search method.

Most of the systems of practical interest, of course, have finite final time, and therefore the optimal control policy for such systems is not given by Equation (12). However, if the magnitude of final time is relatively large compared to time delay, the control policy of Equation (12) represents the structure of the optimal control policy of the time-delay system so well that results very close to optimum can be obtained. This will be illustrated in the numerical examples. Furthermore, in most cases, the direct search on a small number of constant gain matrices [that is, approximating $\mathbf{x}(t+\eta)$ by a small number of straight line segments] yields satisfactory results, thus making this approach particularly attractive from both computational and implementation point of view.

In the examples presented below, the optimum values of the original time-delay system are computed by means of control vector iteration techniques in order to evaluate the proposed methods. Both steepest descent and conjugate gradient methods commonly used for lumped parameter systems can be readily applied to the time-delay system with only minor modifications, as pointed out by Ray and Soliman (1970).

ILLUSTRATIVE EXAMPLES

In order to illustrate and test the two design methods proposed in this paper, we consider three examples: two linear systems and a nonlinear two-stage CSTR system.

Example 1: A Linear System

Let us consider a linear time-delay system described by the scalar equation

$$\dot{x}(t) = x(t) + x(t-\theta) + u(t) \tag{14}$$

$$x(\tau) = 1.0, \quad -\theta \leq \tau \leq 0 \tag{15}$$

with the performance index, to be minimized:

$$J = \int_0^2 (x^2 + u^2)\,dt \tag{16}$$

The same system with $\theta = 1$ was originally presented by Eller et al. (1969) and also considered by Jamshidi and Malek-Zavarei (1972) to illustrate their proposed methods for optimal control of a linear time-delay system. In this example, however, the time delay is taken to be a parameter in the range of $0 \leq \theta \leq 1$ to allow us to investigate the effect of the magnitude of the time delay on the performance of the proposed design methods.

With a Taylor series expansion of the delayed term as in Equation (4), the original time-delay system [Equations (14) and (15)] is simplified to the following nondelay system

$$\dot{x}(t) = \frac{2}{(1+\theta)}\,x(t) + \frac{1}{(1+\theta)}\,u(t) \tag{17}$$

with the initial condition

$$x(o) = 1.0 \tag{18}$$

The optimal control policy for the nondelay system [Equations (17) and (18)] with the performance index of Equation (16) is given by

$$u(t) = -\frac{K(t)}{2(1+\theta)}\,x(t) \tag{19}$$

where $K(t)$ is the solution of the Riccati equation

$$\dot{K}(t) = -\frac{4}{(1+\theta)}\,K(t) + \frac{1}{2(1+\theta)^2}\,K^2(t) - 2 \tag{20}$$

with the final condition

$$K(2) = 0 \tag{21}$$

The control policy given by Equation (19) provides a suboptimal, rather than an optimal, control for the original time-delay system, since approximation of the delayed state by Taylor series expansion has been employed in deriving state equation [Equation (17)]. However, such approximation is good if the time delay is small in magnitude. The values of the performance index for the time-delay system with the suboptimal feedback control policy of Equation (19) are shown against the magnitude of time delay in Figure 1 (dashed line). It is seen that for $\theta < 0.3$, the degree of suboptimality with the proposed design

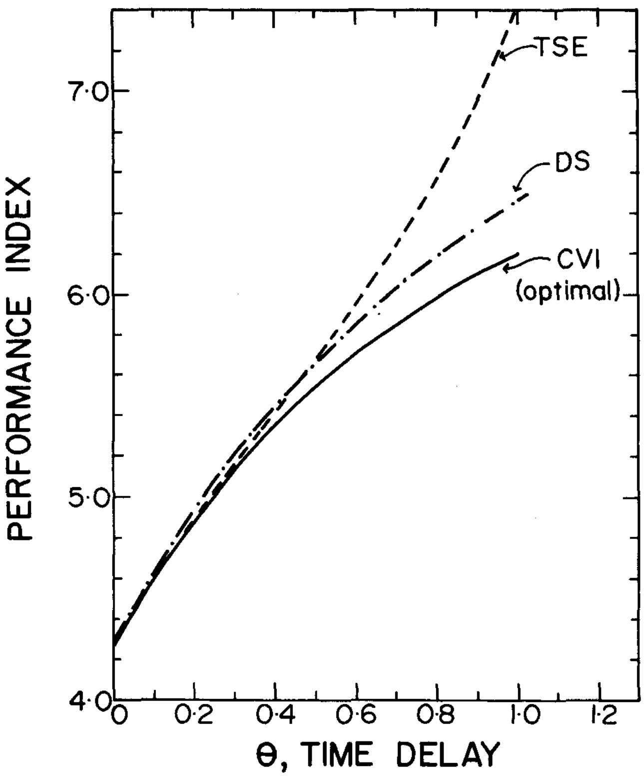

Fig. 1. Degree of suboptimality of the proposed design methods as a function of time delay for $t_f = 2$.

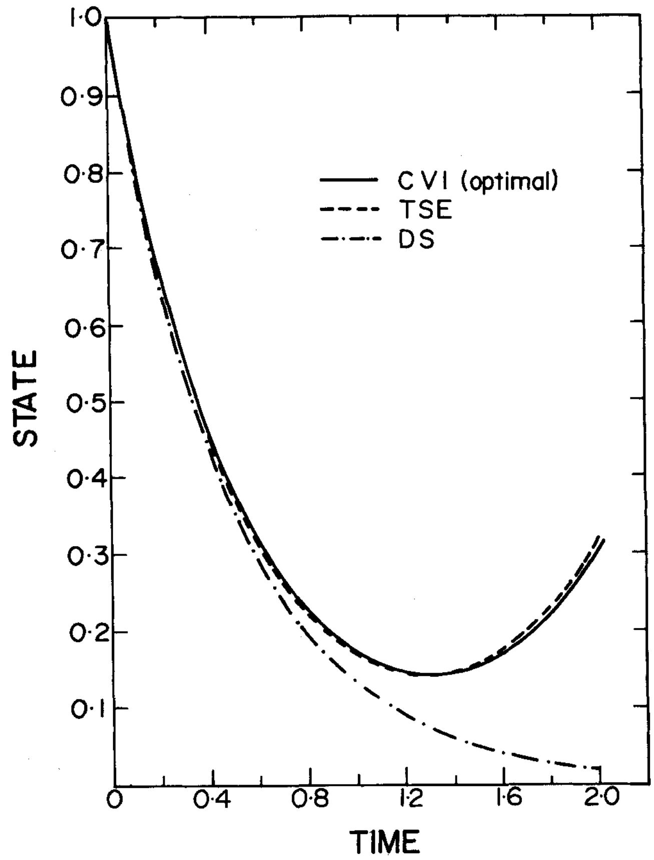

Fig. 2. State trajectories of the time-delay system for $\theta = 0.1$.

TABLE 1. VALUES OF PERFORMANCE INDEX WITH VARIOUS NUMBERS OF DELAYED TERMS IN CONTROL POLICY FOR $\theta = 1$

Number of delayed terms in control policy n	Performance index J
0	6.5645
1	6.4998
2	6.4697
4	6.4649

method is negligible, and thus this approach is very attractive.

In applying the second design method, it is first important to determine how many straight-line segments are necessary for the approximation of the integral in Equation (12) in order to obtain reasonably good results. Therefore, preliminary runs were made for the case of $\theta = 1$ by applying the direct search optimization technique of Luus and Jaakola (1973) for the control policies of the form of Equation (13), each containing different number of delayed terms. The results of these computations, presented in Table 1, show that, in this particular example, including more than two delayed terms in the feedback control structure does not affect the value of the performance index significantly. Thus, all the computations were carried out with only two delayed terms in the control policy. That is

$$u(t) = g_0x(t) + g_1x\left(t - \frac{\theta}{2}\right) + g_2x(t - \theta) \quad (22)$$

where g_0, g_1, and g_2 are constant gains (scalars). The problem then consists of determining the best values for the g_i, a three-dimensional direct search optimization problem.

The results obtained by means of this direct search method are presented in Figure 1, along with the results obtained by using the first proposed approach (Taylor series expansion of the delayed states) and the control vector iteration method (steepest descent). In this and following figures, CVI, TSE, and DS are used to denote control vector iteration, Taylor series expansion, and direct search, respectively. It can be immediately observed that for time delays greater than 0.5, the direct search method

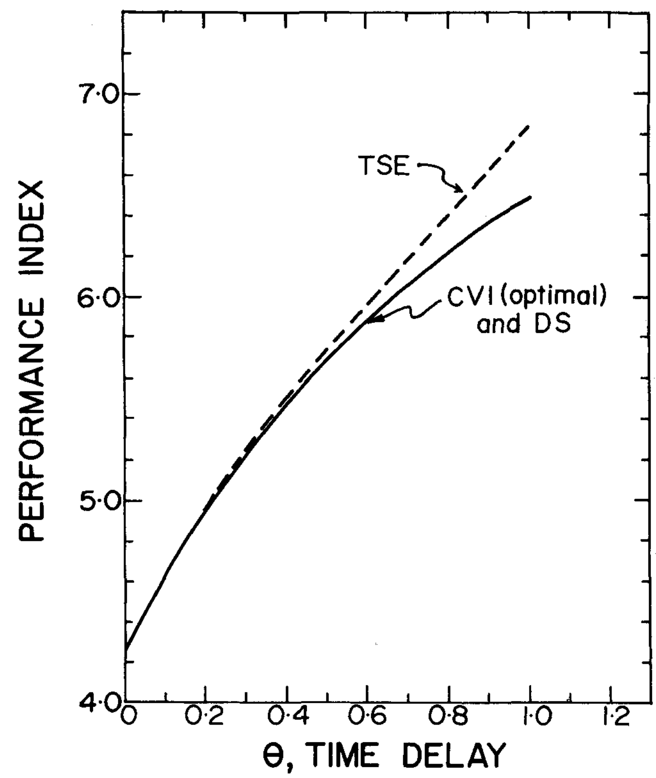

Fig. 3. Degree of suboptimality of the proposed design methods with $t_f = 5$.

Table 2. Comparison of the Performance Index Obtained by the Proposed Design Methods with Optimum for Various Time Delays for Example 2

Time delay θ	Control vector iteration (optimal)	Taylor series expansion	Direct search
0	2.638	2.638	2.638
0.1	2.562	2.562	2.562
0.25	2.526	2.526	2.526
0.5	2.609	2.613	2.609
0.75	2.769	2.782	2.770
1.0	2.932	*	2.932

(Design methods: Control vector iteration, Taylor series expansion, Direct search.)

* Cannot be obtained.

gives better results than the method based on Taylor series expansion. When the time delay is small in magnitude, however, the Taylor series expansion reflects the dynamics of the original time-delay system so well that the resulting state trajectory is almost identical to the optimal state trajectory obtained by the steepest descent method. This is brought out more clearly in Figure 2 for $\theta = 0.1$. Also of interest in Figure 2 is the fact that although the direct search method yields a performance index quite close to the optimum, the resulting state trajectory deviates from the optimal trajectory by approaching the origin near the final time.

In order to investigate the effect of final time on the degree of suboptimality with the proposed design methods, we reexamined the same time-delay system with $t_f = 5$ instead of 2. Here again, the direct search method was based on the control policy of the form of Equation (22). However, the optimal solution was computed by using the conjugate gradient method of Lasdon et al. (1967), since the convergence rate of the steepest descent method was too slow in this problem. The computational results are presented in Figure 3, where the direct search method is shown to give practically the same result as the control vector iteration method. Thus, it can be concluded from Figures 1 and 3 that the direct search method yields good results over a wide range of time delays, while the approach based on Taylor series expansion is superior when time delay is small in magnitude. In particular, if the final time is relatively large compared to the time delay, excellent results can be obtained by the direct search approach with the control policy including only two delayed terms, as demonstrated in Figure 3.

Example 2: A Linear System

Consider a linear time-delay system of the form

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -10\,x_1(t) - 5x_2(t) - 2x_1(t-\theta) \\ &\quad - x_2(t-\theta) + u(t) \end{aligned}\right\} \tag{23}$$

with the initial state profile

$$x_1(\tau) = x_2(\tau) = 1.0, \quad -\theta \leq \tau \leq 0 \tag{24}$$

and the performance index, to be minimized

$$J = \frac{1}{2}\int_0^5 (10\,x_1^2 + x_2^2 + u^2)\,dt \tag{25}$$

The same system with $\theta = 1$ was also considered by Chan and Perkins (1973) to illustrate their proposed parameter imbedding method for computing optimal control of linear time-delay systems.

By approximating both delayed state variables in Equation (23) by Taylor series expansion of the form

$$\left.\begin{aligned} x_1(t-\theta) &\simeq x_1(t) - \theta\,\dot{x}_1(t) \\ x_2(t-\theta) &\simeq x_2(t) - \theta\,\dot{x}_2(t) \end{aligned}\right\} \tag{26}$$

the original time-delay system described by Equations (23) and (24) is reduced to the following nondelay system

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{1}{(1-\theta)}[-12x_1(t) \\ &\quad + (2\theta - 6)\,x_2(t) + u(t)] \end{aligned}\right\} \tag{27}$$

with the initial condition

$$x_1(o) = x_2(o) = 1.0 \tag{28}$$

As illustrated in Example 1, the optimal control for the approximate model given by Equations (27) and (28) can be readily computed from the well-known theory. This control policy then constitutes a suboptimal control for the original time-delay system when applied to the original system. The resulting values of the performance index of the original time-delay system with the suboptimal control are given in Table 2, along with the optimum values obtained by the method of steepest descent. It is seen that this design approach gives satisfactory results over the wide range of time delays, although some degree of suboptimality is involved for the time delays greater than 0.5. It should be noted that, in this example, the result for $\theta = 1$ cannot be obtained by this approach owing to the form of the approximate model of Equation (27).

As in the previous example, the direct search method for this problem is based on the control structure which includes two delayed terms for each delayed state variable; that is

$$u(t) = g_1x_1(t) + g_2x_1\left(t - \frac{\theta}{2}\right) + g_3x_1(t-\theta) + g_4x_2(t) + g_5x_2\left(t - \frac{\theta}{2}\right) + g_6x_2(t-\theta) \tag{29}$$

The computational results obtained by this direct search approach are also shown in Table 2. As can be seen, the direct search method yields practically the same values of the performance index as the optimum values for a very wide range of time delays, namely, $0 \leq \theta \leq 1$.

Example 3: Nonlinear Two-Stage CSTR

The third example consists of a two-stage nonlinear CSTR system with a first-order irreversible chemical reaction. The system is based on the continuous stirred-tank reactor originally modeled by Aris and Amundson (1958) and considered by Lapidus and Luus (1967) for stability studies. The system is described by the differential-difference equations

$$\dot{x}_1(t) = 0.5 - x_1(t) - R_1 = f_1(t) \tag{30}$$

$$\dot{x}_2(t) = -2(x_2(t) + 0.25) - u_1(t)\,[x_2(t) + 0.25] + R_1 = f_2(t) \tag{31}$$

$$\dot{x}_3(t) = x_1(t-\theta) - x_3(t) - R_2 + 0.25 \tag{32}$$

$$\dot{x}_4(t) = x_2(t-\theta) - 2x_4(t) - u_2(t)\,[x_4(t) + 0.25] + R_2 - 0.25 \tag{33}$$

where x_1 and x_3 are normalized concentration variables in tanks 1 and 2, respectively, and x_2 and x_4 are normalized temperature variables in tanks 1 and 2, respectively. The reaction rate terms in tanks 1 and 2 are given by

Table 3. Values of the Minimum Performance Index Obtained by Various Design Methods for the Two-Stage CSTR System

Time daley θ	Design methods: Control vector iteration (optimal)	Taylor series expansion + control vector iteration	Direct search
0.05	0.02297	0.02310	0.02310
0.1	0.02328	0.02328	0.02330
0.2	0.02372	0.02396	0.02407
0.4	0.02476	0.02483	0.02502
0.6	0.02531	0.02560	0.02550
0.8	0.02561	0.02780	0.02606

$$R_1 = [x_1(t) + 0.5] \exp\left(\frac{25\, x_2(t)}{x_2(t) + 2}\right) \quad (34)$$

$$R_2 = [x_3(t) + 0.25] \exp\left(\frac{25\, x_4(t)}{x_4(t) + 2}\right) \quad (35)$$

The normalized controls are bounded by

$$\left.\begin{aligned} -1 \leq u_1 \leq 1 \\ -1 \leq u_2 \leq 1 \end{aligned}\right\} \quad (36)$$

and the initial state profiles for the time-delay system are given by

$$\left.\begin{aligned} &x_1(\tau) = 0.15, \quad -\theta \leq \tau \leq 0 \\ &x_2(\tau) = -0.03, \quad -\theta \leq \tau \leq 0 \\ &x_3(o) = 0.10 \\ &x_4(o) = 0 \end{aligned}\right\} \quad (37)$$

The problem is to find controls $u_1(t)$ and $u_2(t)$ $(0 \leq t \leq 2)$ which minimize the performance index

$$J = \int_0^2 (x_1^2 + x_2^2 + x_3^2 + x_4^2 + 0.1\, u_1^2 + 0.1\, u_2^2)\, dt \quad (38)$$

Expanding both time-delay terms $x_1(t - \theta)$ and $x_2(t - \theta)$ as in Equation (26) and substituting into Equations (32) and (33), we get the following approximate model which consists of a set of ordinary differential equations:

$$\dot{x}_1(t) = f_1(t) \quad (39)$$

$$\dot{x}_2(t) = f_2(t) \quad (40)$$

$$\dot{x}_3(t) = x_1(t) - x_3(t) - \theta f_1(t) - R_2 + 0.25 \quad (41)$$

$$\dot{x}_4(t) = x_2(t) - 2\, x_4(t) - u_2(t)\, [x_4(t) + 0.25] - \theta f_2(t) + R_2 - 0.25 \quad (42)$$

This approximate model can be more easily handled, since abundant literature is available on the optimal control of nonlinear lumped parameter systems. In order to investigate the validity of using the approximate model [Equations (39) through (42)] in the optimal control system design for the original time-delay system, the control vector iteration method (conjugate gradient method) was applied to the approximate model for six different values of time delay. The resulting open-loop control policy was applied to the original time-delay system to obtain the values of the performance index shown in the third column of Table 3. When these values are compared with the corresponding optimum values (shown in the second column of Table 3), which were also obtained by applying the conjugate gradient method to the original system, it is found that the design method based on the Taylor series expansion of delayed states gives satisfactory results for the range of time delays considered here, except for the largest time delay of $\theta = 0.8$. It is noted here that in formulating the time-delay system under consideration, time was normalized with respect to the residence time of one tank, and therefore the magnitude of time delays considered in this example ranges from 5 to 80% of the residence time of one tank.

The second design approach for this problem consists of applying the direct search optimization technique of Luus and Jaakola (1973) based on the following structure for controls u_1 and u_2:

$$u_1(t) = g_{11}x_1(t) + g_{12}x_2(t) + g_{13}x_3(t) + g_{14}x_4(t) + g_{15}x_1(t-\theta) + g_{16}x_2(t-\theta) \quad (43)$$

$$u_2(t) = g_{21}x_1(t) + g_{22}x_2(t) + g_{23}x_3(t) + g_{24}x_4(t) + g_{25}x_1(t-\theta) + g_{26}x_2(t-\theta) \quad (44)$$

The control structure given above is adopted for this nonlinear system by generalizing the result for linear time-delay systems. Such generalization has been successfully employed by Luus (1974) for the optimal feedback control of nonlinear systems without time delays. It can be seen from Equations (43) and (44) that the feedback gain matrix consists of twelve elements which must be determined. The values of the performance index obtained by using the direct search design approach are also presented in Table 3 (last column). The results are quite close to the corresponding optimum values, the maximum devia-

Table 4. Elements of the Feedback Gain Matrices Obtained by Direct Search Approach for the Two-Stage CSTR System

g_{ij}	Time delay, θ: 0.05	0.1	0.2	0.4	0.6	0.8
g_{11}	−0.9083	0.2911	2.1043	−1.3236	−3.2510	−3.0822
g_{12}	22.7885	33.3290	40.7754	16.2356	10.8327	10.6552
g_{13}	−2.1227	1.0250	6.4836	−1.0325	2.6665	4.1195
g_{14}	0.5943	2.6302	1.4645	1.0447	−0.1413	−1.1236
g_{15}	5.7222	2.0111	−2.1991	1.9755	1.4702	1.4986
g_{16}	5.7054	−5.2702	−5.1107	−6.6490	−3.3846	0.3583
g_{21}	5.9420	0.9824	0.0597	−4.6142	−7.3141	−6.3192
g_{22}	12.2235	−0.0129	−2.5586	1.1352	−5.2340	−4.0798
g_{23}	0.0978	8.1001	10.3228	1.3662	6.6302	5.9627
g_{24}	2.9176	10.3833	10.1119	3.1529	5.8102	4.8942
g_{25}	−2.0439	−4.8507	−4.7905	3.0773	0.8892	−0.1845
g_{26}	4.9209	2.7632	6.1912	−5.6067	−6.6215	−10.5583

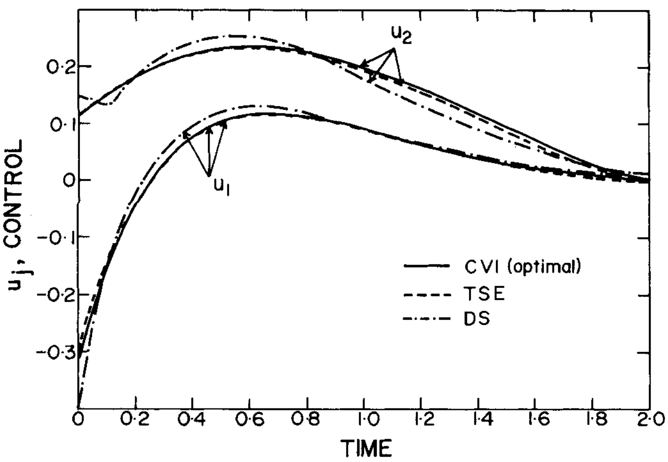

Fig. 4. Comparison of control policies obtained by the proposed design methods with optimal control policy for two-stage CSTR with $\theta = 0.1$.

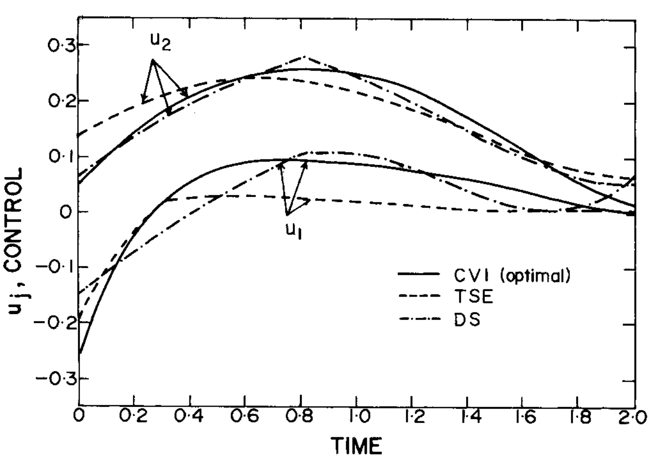

Fig. 5. Comparison of optimal and suboptimal control policies for the two-stage CSTR with $\theta = 0.8$.

tion being 1.7% for $\theta = 0.8$. The elements of the resulting constant feedback gain matrices are given in Table 4.

In order to further illustrate the viability of the proposed design methods, the suboptimal control policies for $\theta = 0.1$ obtained by the Taylor series expansion and the direct search approaches are shown in Figure 4, along with the corresponding optimal control policy. It can be observed that for this small time delay both proposed design approaches not only yield the values of performance index close to the optimum, but also the resulting control policies approximate the optimal control policy very well. In particular, the suboptimal control policy obtained by the Taylor series expansion approach is almost identical to the optimal control policy, making this approach very attractive for the case with small time delays. It is interesting at this point to examine the control policies for a larger value of time delay in order to investigate the effect of increasing time delay. Figure 5 compares the optimal and suboptimal control policies for $\theta = 0.8$, where both suboptimal control policies obtained by the proposed methods are shown to deviate considerably from the optimal control policy. However, the control policy resulting from the direct search approach is such that the deviations from the optimal policy compensate each other, yielding significantly better values of the performance index than the Taylor series expansion approach (1.7% error vs. 8.6%). It is expected that better results can be obtained by means of direct search approach by including in the feedback control structure more delayed terms for each delayed state variable, but at the cost of higher dimensionality in the resulting optimization problem.

DISCUSSION

The examples presented in this paper show that the direct search approach yields satisfactory results over a wide range of time delays, while the Taylor series expansion approach demonstrates its superiority over the direct search approach for sufficiently small time delays. However, the degree of suboptimality with the Taylor series expansion approach for a time delay of interest cannot be determined a priori, and thus it may be desirable to apply both proposed methods to a specific problem at hand and compare the results before deciding which method to use for on-line control.

From the actual implementation point of view, the direct search approach is more attractive, since the resulting control system is based on a constant feedback gain matrix, and this approach yields particularly good results when the final time is relatively large compared to the time delay, the situation commonly encountered in practice. Although the Taylor series expansion approach leads to the control law containing only the present state, the feedback gain matrix is time varying, and therefore the implementation of the control system may present some difficulties.

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NOTATION

$\mathbf{A}$ = constant coefficient matrix for state in Equation (1)
$\mathbf{B}$ = constant coefficient matrix for delayed state in Equation (1)
$\mathbf{C}$ = constant coefficient matrix for control in Equation (1)
$f_1(t)$ = quantity given in Equation (30)
$f_2(t)$ = quantity given in Equation (31)
$\mathbf{G}_i$ = constant feedback gain matrix
g_{ij} = elements of $\mathbf{G}_i$
$\mathbf{I}$ = identity matrix
J = performance index
$\mathbf{K}(t)$ = time varying feedback gain matrix given by Equations (10) and (11)
$\mathbf{K}_0$ = constant feedback gain matrix in Equation (12)
$\mathbf{K}_1$ = time varying feedback gain matrix in Equation (12)
$\mathbf{M}$ = matrix defined by Equation (6)
$\mathbf{N}$ = matrix defined by Equation (7)
n = number of delayed terms in control policy
$\mathbf{Q}$ = constant symmetric positive semidefinite matrix
$\mathbf{R}$ = constant symmetric positive definite matrix
R_1 = reaction rate in tank 1 given by Equation (34)
R_2 = reaction rate in tank 2 given by Equation (35)
t = time
t_f = final time
$\mathbf{x}(t)$ = state vector
$x_i(t)$ = state variable
$\mathbf{u}(t)$ = control vector
θ = time delay
$\boldsymbol{\varphi}$ = initial state profile for a time-delay system

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Transient Diffusion in Solids with a Bipore Distribution

Transient diffusion of *n*-butane, iso-butane, and 1-butene in synthetic CaX(Na) spherical pellets is studied in a constant volume, well-stirred system. The intracrystalline diffusion coefficients are in the order of 10^{-14} cm^2/s, and the activation energy is about 7.0 Kcal/g-mole. Experimental data agree well with a mathematical model developed to describe transient diffusion.

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SCOPE

The diffusion in porous pellets formed from zeolite crystals usually involves two main mechanisms: diffusion through the micropore structure within the crystals (intracrystalline diffusion) and diffusion through macropore structure between crystals (intercrystalline diffusion). The determination of each individual diffusion coefficient can be of importance in data interpretation and improving engineering design in operations such as purification, separation, and catalytic conversions.

Previous work on the study of diffusion in zeolites concentrated mostly in micropore diffusion with powder forms. Recent studies on the diffusion in solids with a bipore distribution include the theoretical and experimental investigation of diffusion in ion exchanged resins by Ruckenstein et al. (1971), the experimental study of diffusion of carbon dioxide in commercial 5A pellets by Sargent and Whitford (1971), and mathematical and experimental investigation of diffusion in Linde 13X by Ma and Ho (1974). All the above-mentioned studies were performed under constant pressure conditions. The present work employed a constant volume technique to examine quantitatively the diffusion of C_4-hydrocarbons in micropores. Diffusion coefficients were evaluated based on a mathematical model taking into account intercrystalline and intracrystalline diffusion and adsorption under constant volume conditions. Rates of sorption of *n*-butane, iso-butane, and 1-butene in synthetic CaX(Na) were measured at three temperatures, and the activation energies for diffusion were determined from an Arrhenius plot.

CONCLUSIONS AND SIGNIFICANCE

Micropore diffusion coefficients were determined by assuming that the transport processes in macropores and micropores obey Fick's equation. The activation energy for diffusion was found to be about 6 to 8 Kcal/g-mole for the C_4-hydrocarbons used in the study. The relatively low activation energy may, in part, be due to the large openings of the zeolite X.

A mathematical model was developed to describe the diffusion in a solid with a bipore distribution. The present model differs from previous studies in that it deals with a well-stirred system where the total quantity of the diffusing species is finite. Agreement between experimental data and theoretical results was good. It should be noted that the present model can be reduced to those developed